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Monopole bundles over fuzzy complex projective spaces

Ursula Carow-Watamura[a], Harold Steinacker[b,*], Satoshi Watamura[a]

[a] *Department of Physics, Graduate School of Science, Tohoku University, Aoba-ku, Sendai 980-8578, Japan*
[b] *Institut für Theoretische Physik, Ludwig-Maximilians Universität München, Theresienstr. 37, D-80333 München, Germany*

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Abstract

We give a construction of the monopole bundles over fuzzy complex projective spaces as projective modules. The corresponding Chern classes are calculated. They reduce to the monopole charges in the $N \to \infty$ limit, where N labels the representation of the fuzzy algebra.

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* Corresponding author. Tel.: +49 89 2180 4376; fax: +49 892 2180 4186.
E-mail addresses: ursula@tuhep.phys.tohoku.ac.jp (U. Carow-Watamura), harold.steinacker@physik.uni-muenchen.de (H. Steinacker), watamura@tuhep.phys.tohoku.ac.jp (S. Watamura).

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1. Introduction

Gauge theory on noncommutative spaces has attracted considerable interest in recent years. In particular, topologically nontrivial solutions such as instantons were found on the noncommutative plane. They are conveniently described in terms of projective modules over the algebra $\mathcal{A}$ of functions on the noncommutative space. Additional motivation is provided by the appearance of noncommutative gauge theory as an effective theory of D-branes in string theory. This led to an interpretation of the nontrivial solutions of gauge theory on noncommutative flat space in terms of nonperturbative configurations in the D-brane background, see e.g. [1].

On the other hand, gauge theory on non-flat noncommutative spaces is not very well understood. A particularly nice class of such spaces are the so-called fuzzy spaces, the simplest example being the fuzzy sphere [2]. Understanding field theory and in particular gauge theory on fuzzy spaces is important for several reasons. First, the fuzzy spaces provide a nice regularization of field theory, because they admit only finitely many degrees of freedom. This provides an alternative to lattice regularization, with the advantage of preserving a large symmetry group [3,4]. While much work has been done for the fuzzy sphere, the higher-dimensional spaces such as fuzzy $\mathbb{C}P^n$ are largely unexplored.

The fuzzy spaces also arise in string theory. For example, the fuzzy sphere and its q-deformed version appear as a D-brane in the $SU(2)$ WZW model, as discussed by several authors [5,6]; see also [7] and references therein. In fact, many fuzzy spaces investigated so far can be considered as D-branes on group manifolds [8]. Moreover, gauge theory on the fuzzy sphere appears as an effective theory of the D-branes in S^3, in the $k \to \infty$ limit of the $SU(2)_k$ WZW model at level k. Furthermore, fuzzy spaces also arise as solutions of the IKKT matrix model [9,10]. It is therefore natural to ask for a proper geometrical description and interpretation of such a system, in particular of the topologically nontrivial configurations.

Topological aspects of field theory on the fuzzy sphere have first been discussed in [11]. The formulation as projective modules has been elaborated explicitly for the fuzzy sphere in [12–14], and an alternative approach using matrix models was given in [15].

In Ref. [16,17] the authors have investigated the Dirac operators on the fuzzy sphere which have been proposed in [11,18,19] with respect to their differences, and their relation to topologically nontrivial configurations and the fermion doubling problem have been studied in [20,21].

For physical applications, it is clearly desirable to consider spaces of dimension 4 and higher. The simplest higher-dimensional fuzzy spaces are fuzzy $\mathbb{C}P^2$ and $\mathbb{C}P^n$, which have been studied in [4,22–24].

In this paper, we consider fuzzy $\mathbb{C}P^2$ and $\mathbb{C}P^n$ in more detail, and we present a simple formulation of monopole bundles on fuzzy $\mathbb{C}P^n$ using projective modules. We also introduce a suitable differential calculus, which allows to compute the canonical connection and field strength explicitly. The corresponding Chern classes are calculated. As in the case of the fuzzy sphere, the Chern numbers are integers only in the commutative limit. For related work on a fuzzy four-sphere see [25].

The outline of this paper is as follows. In Section 2, the geometry of classical $\mathbb{C}P^n$ is formulated using two different approaches. The first is in terms of (co)adjoint orbits of $su(n+1)$, and the second using a generalized Hopf fibration. Both lead to a useful characterization in terms of $(n+1)\times(n+1)$ matrices satisfying a quadratic characteristic equation. For better readability we first present the case of $\mathbb{C}P^2$, and then the general case for $\mathbb{C}P^n$ separately.

In Section 3, the fuzzy spaces $\mathbb{C}P^2_N$ and $\mathbb{C}P^n_N$ are discussed from these two points of view, leading again to a quadratic characteristic equation for algebra-valued $(n+1)\times(n+1)$ matrices. This encodes the commutation relations of the coordinate algebra in a compact way. In particular, the Hopf fibration is quantized in terms of a Fock space representation.

In Section 4, we give a construction of the projective modules corresponding to monopole bundles for fuzzy $\mathbb{C}P^n_N$. This is done using an explicit form of the projection operators. We also explain how a section of the constructed line bundles corresponds to a complex scalar field. A similar construction for the classical case of $S^3\to S^2$ has been given by Landi in [26,27].

In Section 5, a differential calculus is constructed, which in the fuzzy case involves more degrees of freedom than in the classical case. This is again typical for fuzzy spaces, and we explain in what sense the classical calculus is recovered in the commutative limit. This calculus is then used to compute the field strength and Chern class for the monopole bundles. We show that the usual (integer) Chern numbers are recovered in the limit of large N.

2. The geometry of $\mathbb{C}\ P^n$

We will discuss two descriptions of $\mathbb{C}P^n$ here. The first is in terms of (co)adjoint orbits of $su(n+1)$, and the second is based on the generalized Hopf fibration $U(1)\to S^{2n+1}\to \mathbb{C}P^n$. Both are manifestly covariant under $SU(n+1)$, which is maintained in their quantization as fuzzy $\mathbb{C}P^n$.

2.1. Adjoint orbits

In general, an adjoint orbit of a (finite-dimensional) matrix Lie group G with Lie algebra $\mathfrak{g}$ is given in terms of some $t\in\mathfrak{g}$ as

$$\mathcal{O}(t)=\{gtg^{-1}; g\in G\}\subset\mathfrak{g}. \tag{1}$$

Then $\mathcal{O}(t)$ can be viewed as a homogeneous space:

$$\mathcal{O}(t)\cong\frac{G}{K_t}. \tag{2}$$

Here $K_t=\{g\in G:[g,t]=0\}$ is the stabilizer of t, which determines the nature of the space $\mathcal{O}(t)$. Any such conjugacy class is invariant under the adjoint action of G. “Regular” conjugacy classes are those with K_t being the maximal torus, and have maximal dimension $\dim(\mathcal{O}(t))=\dim(G)-\mathrm{rank}(G)$. Here we are interested in degenerate orbits such as $\mathbb{C}P^2$ or $\mathbb{C}P^n$. They correspond to degenerate t, and have dimension $\dim(\mathcal{O}(t))=\dim(G)-\dim(K_t)$.

A nice way to characterize the type of the orbit (for matrix Lie groups) is through its characteristic equation $\chi(Y) = 0$ for $Y \in \mathcal{O}(t)$, which is invariant under conjugation and therefore depends only on the eigenvalues of t.

For example, in order to obtain $\mathbb{C}P^2 = SU(3)/(SU(2) \times U(1))$, we should choose the matrix $t \in \mathfrak{g} \cong \{Y \in Mat(3, \mathbb{C}); Y^\dagger = Y, \mathrm{tr}(Y) = 0\}$ with only two distinct eigenvalues. A natural choice for an element of the adjoint orbit is hence $t = \mathrm{diag}(-1, -1, 2)$, so that the stabilizer is $K_t = SU(2) \times U(1)$. The normalization of the matrix t defines the scale of the resulting $\mathbb{C}P^2$, which is irrelevant for the discussion here. [1]

The characteristic equation for $\mathbb{C}P^2$ is therefore quadratic, and has the form

$$\chi(Y) = (Y + 1)(Y - 2) = 0. \tag{3}$$

This equation characterizes $\mathbb{C}P^2$ as submanifold in the embedding space $\mathbb{R}^8$. We will see that an analogous characteristic equation holds for fuzzy $\mathbb{C}P^n$.

This construction of $\mathbb{C}P^2$ can also be understood as follows: the 3×3 matrix

$$P = \tfrac{1}{3}(Y + 1) \in Mat(3, \mathbb{C}) \tag{4}$$

satisfies

$$P^2 = P, \qquad \mathrm{Tr}(P) = 1 \tag{5}$$

as a consequence of (3), hence $P \in Mat(3, \mathbb{C})$ is a projector of rank 1 and can be written as

$$P = |z^i\rangle\langle z^i| = (z^i)^\dagger(z^i),$$

where $(z^i) = (z^1, z^2, z^3) \in \mathbb{C}^3$ is normalized as $\langle z^i|z_i\rangle = 1$. Such projectors are equivalent to rays in $\mathbb{C}^3$, which give the second description of $\mathbb{C}P^2$ as $S^5/U(1)$. The adjoint action on Y corresponds to the fundamental representation on $\mathbb{C}^3$.

Similarly, to obtain $\mathbb{C}P^n \cong SU(n + 1)/(SU(n) \times U(1))$, we need a matrix $t \in su(n + 1)$ with two distinct eigenvalues and multiplicities $(n, 1)$, hence a natural choice is $t = \mathrm{diag}(-1, -1, \ldots, -1, n)$ up to normalization. It satisfies the characteristic equation

$$\chi(Y) = (Y + 1)(Y - n) = 0. \tag{6}$$

Again, this can be understood by considering the $(n + 1) \times (n + 1)$ matrix

$$P = \frac{1}{n + 1}(Y + 1) \in Mat(n + 1, \mathbb{C}), \tag{7}$$

which is also a projector of rank 1. Hence $P \in Mat(n + 1, \mathbb{C})$ can be written as

$$P = |z^i\rangle\langle z^i|,$$

where $\langle z^i|z_i\rangle = 1$. This is the relation to the second description of $\mathbb{C}P^n$ as $S^{2n+1}/U(1)$.

[1] Notice that there is also the "conjugated" space with $t \cong \mathrm{diag}(1, 1, -2)$, which correspond to the charge conjugation of the $\mathbb{C}P^2$ defined here.

2.2. Global coordinates for $\mathbb{C}P^2$ and $\mathbb{C}P^n$

For later purpose, we introduce coordinates on $\mathbb{C}P^2$ and $\mathbb{C}P^n$. It is useful to choose an overcomplete set of global coordinates, which is easily generalized to the fuzzy case. Let us first consider $\mathbb{C}P^2$, described as above by the matrix $Y = g^{-1}tg \in Mat(3, \mathbb{C})$. It is natural to write the matrix Y in terms of the Gell–Mann matrices λ_a of $su(3)$ as

$$Y = y_a\lambda_a. \tag{8}$$

The Gell–Mann matrices satisfy

$$\mathrm{tr}(\lambda_a\lambda_b) = 2\delta_{ab}, \qquad \lambda_a\lambda_b = \tfrac{2}{3}\delta_{ab} + (i f_{abc} + d_{abc})\lambda_c, \tag{9}$$

where f_{abc} are the totally antisymmetric structure constants, and d_{abc} the totally symmetric invariant tensors of $su(3)$. The λ_a are related to the generators T_a of the Lie algebra via

$$\lambda_a = 2\pi_{\Lambda_{(1)}}(T_a). \tag{10}$$

Here $\pi_{\Lambda_{(1)}}$ denotes the fundamental representation of $su(3)$ with highest weight $\Lambda_{(1)}$. The characteristic equation

$$Y^2 = Y + 2 \tag{11}$$

written in terms of the coordinates y_a in Eq. (8) takes the form

$$g_{ab}y_a y_b = 3, \qquad d_{abc}y_a y_b = y_c. \tag{12}$$

It is clear from the above construction that this set of relations indeed characterizes the appropriate adjoint orbit in $su(3)$.

For $\mathbb{C}P^n$, we consider the generalized Gell–Mann matrices of $su(n+1)$ which are defined by

$$\lambda_a = 2\pi_{\Lambda_{(1)}}(T_a), \tag{13}$$

where T_a are the generators of $su(n+1)$. They satisfy

$$\lambda_a\lambda_b = \frac{2}{n+1}\delta_{ab} + (i f_{abc} + d_{abc})\lambda_c. \tag{14}$$

Then the characteristic equation

$$Y^2 = (n-1)Y + n \tag{15}$$

using the expansion $Y = y_a\lambda_a = g^{-1}tg$ takes the form

$$g_{ab}y_a y_b = \frac{n(n+1)}{2}, \qquad d_{abc}y_a y_b = (n-1)y_c. \tag{16}$$

2.2.1. Some geometry

Notice that the symmetry group $SU(3)$ contains both "rotations" as well as "translations". The generators J_a act on a point $Y = y_a\lambda_a \in \mathbb{C}P^2$ as

$$J_aY = \tfrac{1}{2}[\lambda_a, Y] = \tfrac{1}{2}y_b[\lambda_a, \lambda_b] = i f_{abc}y_b\lambda_c. \tag{17}$$

In terms of the coordinate functions on the embedding space $\mathbb{R}^8$, this can be realized as differential operator

$$J_a = \frac{i}{2} f_{abc}(y_b \partial_c - y_c \partial_b). \tag{18}$$

Now we can identify the rotations in $SU(3)$: Consider the "south pole" on $\mathbb{C}P^2$, for $Y = \mathrm{diag}(-1,-1,2) = -r\lambda^8$ for the radius $r = \sqrt{3}$, hence $y_a = -r\delta_{a,8}$. The rotation subgroup is generated by its stabilizer algebra $su(2) \times u(1)$, which is generated by the elements

$$\{\lambda_1, \lambda_2, \lambda_3, \lambda_8\} \tag{19}$$

of $su(3)$ (using the standard conventions). It is a subgroup of the Euclidean rotation group $SO(4) = SU(2)_L \times SU(2)_R$. The remaining generators

$$\{\lambda_4, \lambda_5, \lambda_6, \lambda_7\} \tag{20}$$

change the position of $Y = -r\lambda_8 \in \mathbb{C}P^2$, hence they correspond to "translations". All this generalizes to $\mathbb{C}P^n$ in an obvious way.

There is another interesting subspace of $\mathbb{C}P^2$: the "north sphere". This is a nontrivial cycle of $\mathbb{C}P^2$ which will be useful later. Consider again the parameterization of $\mathbb{C}P^2$ in terms of 3×3 matrices $Y = U^{-1}\mathrm{diag}(-1,-1,2)U$ introduced in Section 2.1. Using a suitable $U \in SU(3)$, we can put it into the form

$$Y = \left(\begin{array}{c|c} \frac{1}{2} + y_i\sigma^i & 0 \\ \hline 0 & -1 \end{array}\right). \tag{21}$$

This is the subspace of $\mathbb{C}P^2$ with maximal value of $y_8 = \frac{1}{2}r$, where $y_{4,5,6,7} = 0$ and $2r^2 = \mathrm{Tr}\, Y^2 = 2\sum_{a=1,2,3,8} y_a^2$. It follows that $y_1^2 + y_2^2 + y_3^2 = \frac{3}{4}r^2$, which is a sphere of radius $\frac{\sqrt{3}}{2}r$.

A similar sphere can be found for all $\mathbb{C}P^n$: consider again matrices of the form (21) with $n-1$ entries -1 in the lower right block. Then the upper left block has the form $\frac{n-1}{2} + y_i\sigma^i$, which has eigenvalues $(-1, n)$ provided $y_1^2 + y_2^2 + y_3^2 = \frac{n+1}{2n}r^2$, hence it is a sphere of radius $\sqrt{\frac{n+1}{2n}}r$. Here r is the radius of $\mathbb{C}P^n$. One can now choose a Gell–Mann basis of $su(n+1)$ which contain the above σ^i, so that all other $\mathrm{d}y_a$ for $a \neq 1,2,3$ vanish on this sphere. This implies that these are non-trivial cycles, see Section 5.3.

2.3. Harmonic analysis

Fuzzy $\mathbb{C}P^n$ is defined as a particular (finite) noncommutative algebra which is covariant under $SU(3)$, and is interpreted as quantization of the algebra of functions of $\mathbb{C}P^n$. To justify the construction, we first have to understand the space of harmonics on $\mathbb{C}P^n$, and then compare it with the noncommutative case. This can be done using the Hopf fibration $U(1) \to S^{2n+1} \to \mathbb{C}P^n$.

First, consider the space of equivariant functions over S^{2n+1}. Define the $U(1)$ action on $\mathbb{C}^{n+1}$ as

$$\omega \circ (z^i, \overline{z}_i) = (z^i\omega, \overline{z}_i\overline{\omega}), \tag{22}$$

where $\omega \in U(1)$. With this $U(1)$ action, we define the equivariant functions $C(\kappa, \mathbb{C}^{n+1})$ of $z^i, \overline{z}_j$ by[2]

$$C(\kappa, \mathbb{C}^{n+1}) = \{f \in \mathrm{Pol}(z^i, \overline{z}_j), i, j = 1, \ldots, n+1 \,\text{and}\, \omega \circ f = f\omega^\kappa, \}$$
$$= \bigoplus_p \{\mathrm{Pol}_{p,q}(z, \overline{z}), p - q = \kappa\}, \tag{23}$$

where $\mathrm{Pol}_{p,q}(z, \overline{z})$ denotes the polynomial functions of degree (p, q) in the coordinates z^i resp. $\overline{z}_j$ on $\mathbb{C}^{n+1}$. As a representation of $su(n+1)$, it has the structure

$$\mathrm{Pol}_{p,q}(z, \overline{z}) \cong V_{(p,0,\ldots,0)} \otimes V_{(0,\ldots,0,q)},$$

where $V_{(d_1,\ldots,d_n)}$ denotes the highest weight irrep (=irreducible representation) with Dynkin indices $(d_1, \ldots, d_n)$.

In order to identify these with the functions on S^{2n+1}, we impose the condition $r^2 = \sum_i z^i \overline{z}_i$, which defines the equivariant functions $C(\kappa, S^{2n+1})$. By construction, it is clear that $C(0, S^{2n+1})$ is isomorphic to the space of functions $C(\mathbb{C}P^n)$ on $\mathbb{C}P^n$. Since $V_{(p,0,\ldots,0)} \otimes V_{(0,\ldots,0,p)} = \oplus_{n=0}^{p} V_{(n,0,\ldots,0,n)}$ and taking the radius r into account, it follows that $C(0, S^{2n+1})$ decomposes under $su(n+1)$ as

$$C(\mathbb{C}P^n) \cong C(0, S^{2n+1}) = \bigoplus_{p=0}^{\infty} V_{(p,0,\ldots,0,p)}. \tag{24}$$

Similarly, $C(\kappa, S^{2n+1})$ can be identified with the space of sections $\Gamma_\kappa(\mathbb{C}P^n)$ of the line bundle on $\mathbb{C}P^n$ with monopole number κ. Moreover, there is a natural multiplication of two equivariant polynomials $C(\kappa, S^{2n+1})$ and $C(\kappa', S^{2n+1})$ such that

$$C(\kappa, S^{2n+1}) \times C(\kappa', S^{2n+1}) \longrightarrow C(\kappa + \kappa', S^{2n+1}). \tag{25}$$

Therefore $\Gamma_\kappa(\mathbb{C}P^n) \cong C(\kappa, S^{2n+1})$ is a module over $C(\mathbb{C}P^n)$. The decomposition under $su(n+1)$ is similar to (24):

$$\Gamma_\kappa(\mathbb{C}P^n) \cong C(\kappa, S^{2n+1}) = \bigoplus_{n=0}^{\infty} V_{(n,0,\ldots,0,n+\kappa)}, \tag{26}$$

see also [4]. We will recover this structure of harmonics in the fuzzy case, up to some cutoff.

3. Fuzzy complex projective spaces $\mathbb{C}\,P_N^n$

In general, (co)adjoint orbits (1) on G can be quantized in terms of a simple matrix algebra $End_{\mathbb{C}}(V_N)$, where V_N are suitable representations of G. The appropriate representations V_N can be identified by matching the spaces of harmonics (i.e. using harmonic analysis), see [8] for the general case. Fuzzy $\mathbb{C}P^2$ has been introduced in [4,22], and fuzzy $\mathbb{C}P^n$ in [23].

[2] If we follow strictly the Hopf fibration, we should first impose the $SU(n+1)$ invariant condition $\sum z^i \overline{z}_i = 1$ and consider functions over S^{2n+1}. However we impose this constraint later, which is more appropriate in the fuzzy case.

3.1. $\mathbb{C}P_N^2$

Again we first consider $\mathbb{C}P^2$. To identify the correct representations V_N of $su(3)$, we must match the space of harmonics (24) with the decomposition of $End_{\mathbb{C}}(V_N)$ under the adjoint, which is

$$End(V_N) = V_N \otimes V_N^* = \bigoplus_\lambda n_\lambda V_\lambda \tag{27}$$

for certain multiplicities n_λ. Here λ denotes the highest weight. It is easy to see that for both $V_N = V_{(N,0)}$ and $V'_N = V_{(0,N)} = V_N^*$, we have

$$V_N \otimes V_N^* = V_{(N,0)} \otimes V_{(0,N)} \cong \bigoplus_{p=0}^{N} V_{(p,p)}, \tag{28}$$

which matches (24) up to a cutoff. Therefore we define fuzzy $\mathbb{C}P^2$ as

$$\mathbb{C}P_N^2 := End_{\mathbb{C}}(V_N) = Mat(D_N, \mathbb{C}) \tag{29}$$

for $V_N = V_{(N,0)}$,[3] where

$$D_N = \dim(V_N) = \frac{(N+1)(N+2)}{2}. \tag{30}$$

Under the (adjoint) action of $su(3)$, it decomposes into the harmonics (28) $\oplus_{p=0}^N V_{(p,p)}$, cp. [4]. Comparing with (24), these harmonics are in one to one correspondence with the harmonics on classical $\mathbb{C}P^2$ up to the cutoff at $p = N$. The remarkable point is that this finite space of harmonics closes under the matrix multiplication in $\mathbb{C}P_N^2$. Hence by construction, fields on $\mathbb{C}P^2$ can be approximated by $\mathbb{C}P_N^2 = Mat(D_N, \mathbb{C})$, therefore field theory on fuzzy $\mathbb{C}P_N^2$ should be a good regularization for field theory on $\mathbb{C}P^2$.

To make the correspondence with classical $\mathbb{C}P^2$ more explicit, we consider the $3D_N \times 3D_N$ matrix

$$X = \sum_a \xi_a \lambda_a, \tag{31}$$

where λ_a are the Gell–Mann matrices as before, and

$$\xi_a = \pi_{V_N}(T_a) \in \mathbb{C}P_N^2 \tag{32}$$

denotes the representation of $T_a \in su(3)$ on V_N. The coordinate functions $x_a = (x_1, \ldots, x_8)$ on fuzzy $\mathbb{C}P^2$ are defined by

$$x_a = \Lambda_N \xi_a \in \mathbb{C}P_N^2. \tag{33}$$

They are operators acting on V_N. Here Λ_N is a scaling parameter which will be fixed below. By construction, the x_a transform in the adjoint under $su(3)$, just like the classical coordinate functions y_a introduced in Section 2.2.

[3] Alternatively, $\mathbb{C}P_N^{2*}$ for $V'_N = V_{(0,N)}$, which is equivalent as algebra.

To find the relations among these generators x_a, we can use the characteristic equation (A.7) of X given in Appendix A:

$$X^2 = \xi_a \xi_b \left(\frac{2}{3}\delta_{ab} + (i f_{abc} + d_{abc})\lambda_c \right) = \frac{2}{3}\left(\frac{1}{3}N^2 + N \right) + \left(\frac{N}{3} - 1 \right) X. \tag{34}$$

Using $f_{abc} f_{dbc} = 3\delta_{ad}$, we obtain

$$i f_{abc}\xi_a \xi_b = -\tfrac{3}{2}\xi_c, \quad [\xi_a, \xi_b] = i f_{abc}\xi_c, \tag{35}$$

$$g_{ab}\xi_a \xi_b = (\tfrac{1}{3}N^2 + N), \tag{36}$$

$$d_{abc}\xi_a \xi_b = \left(\frac{N}{3} + \frac{1}{2} \right) \xi_c. \tag{37}$$

Hence taking the scale parameter Λ_N to be

$$\Lambda_N = \frac{1}{\sqrt{\frac{1}{3}N^2 + N}}, \tag{38}$$

we find the defining relation of the algebra $\mathbb{C}P^2_N$:

$$[x_a, x_b] = \frac{i}{\sqrt{\frac{1}{3}N^2 + N}} f_{abc} x_c, \tag{39}$$

$$g_{ab} x_a x_b = 1, \tag{40}$$

$$d_{abc} x_a x_b = \frac{\frac{N}{3} + \frac{1}{2}}{\sqrt{\frac{1}{3}N^2 + N}} x_c \tag{41}$$

in agreement with Balachandran et al. [22].

Let us verify that $\mathbb{C}P^2_N$ admits an approximate "south pole" at $x_8 \approx -1$. It corresponds to the lowest weight state of $V_{(N,0)}$ which has eigenvalue $\xi_8 = -\frac{2N}{\sqrt{3}}$, hence $x_8 = -\frac{N}{\sqrt{N^2+3N}} \approx -1$. For $\mathbb{C}P^{2*}_N$, one would obtain an approximate north pole, but no south pole.

The "angular momentum" operators (generators of $SU(3)$) now become inner derivations:

$$J_a f(x) = [\xi_a, f] \tag{42}$$

and $J_a x_b = [\xi_a, x_b] = i f_{abc} x_c$, as classically. Recall that the symmetry group $SU(3)$ contains both "rotations" as well as "translations".

The integral on $\mathbb{C}P^2_N$ is given by the suitably normalized trace:

$$\int f(x) = \frac{1}{D_N}\mathrm{Tr}(f) = \frac{2}{(N+1)(N+2)}\mathrm{Tr}(f), \tag{43}$$

which is clearly invariant under $SU(3)$.

3.2. $\mathbb{C}\ P^n_N$

The construction for fuzzy $\mathbb{C}P^n$ is entirely analogous to fuzzy $\mathbb{C}P^2$. To identify the correct representations V_N of $su(n+1)$, we must match the space of harmonics (24) with the decomposition of $End(V_N) = V_N \otimes V_N^* = \oplus_\lambda n_\lambda V_\lambda$ for certain multiplicities n_λ. Similar to the case of $su(3)$, it is easy to show that

$$V_N \otimes V_N^* \cong \bigoplus_{p=0}^{N} V_{(p,0,\ldots,0,p)}, \tag{44}$$

where

$$V_N := V_{(N,0,\ldots,0)}.$$

The representations appearing in the r.h.s. of Eq. (44) match (24) up to the cutoff N. We therefore define the algebra of the fuzzy projective space by

$$\mathbb{C}P^n_N := End_{\mathbb{C}}(V_N) = Mat(D_N, \mathbb{C}), \tag{45}$$

where

$$D_N = \dim(V_N) = \frac{(N+n)!}{n!N!} \tag{46}$$

from Weyl's dimension formula. The fuzzy coordinates and their commutation relations are obtained again by considering the $(n+1)D_N \times (n+1)D_N$ matrix

$$X = \sum_a \xi_a \lambda_a, \tag{47}$$

where λ_a are the Gell–Mann matrices of $su(n+1)$ and

$$\xi_a = \pi_{V_N}(T_a) \in \mathbb{C}P^n_N. \tag{48}$$

The coordinate functions x_a for $a = 1, \ldots, n^2+2n$ on fuzzy $\mathbb{C}P^n$ are defined by

$$x_a = \Lambda_N \xi_a \in \mathbb{C}P^n_N. \tag{49}$$

Λ_N is a scaling parameter which will be fixed below. By construction, the x_a transform in the adjoint under $su(n+1)$, just like the classical coordinate functions y_a introduced in Section 2.2. Using the characteristic Eq. (A.11) of X:

$$\left(X - \frac{nN}{n+1}\right)\left(X + \frac{N}{n+1} + 1\right) = 0, \tag{50}$$

we have

$$\begin{aligned} X^2 &= \xi_a \xi_b \left(\frac{2}{n+1}\delta_{ab} + (i f_{abc} + d_{abc})\lambda_c\right) \\ &= \frac{n}{n+1}\left(\frac{1}{n+1}N^2 + N\right) + \left(\frac{N(n-1)}{n+1} - 1\right) X. \end{aligned} \tag{51}$$

Using $f_{abc}f_{dbc} = (n+1)\delta_{ad}$, we obtain

$$if_{abc}\xi_a\xi_b = -\frac{n+1}{2}\xi_c, \quad [\xi_a, \xi_b] = if_{abc}\xi_c, \tag{52}$$

$$g_{ab}\xi_a\xi_b = \frac{n}{2}\left(\frac{1}{n+1}N^2 + N\right), \tag{53}$$

$$d_{abc}\xi_a\xi_b = (n-1)\left(\frac{N}{n+1} + \frac{1}{2}\right)\xi_c. \tag{54}$$

Hence for

$$\Lambda_N = \frac{1}{\sqrt{\frac{n}{2(n+1)}N^2 + \frac{n}{2}N}}, \tag{55}$$

we find

$$[x_a, x_b] = i\Lambda_N f_{abc}x_c, \tag{56}$$

$$g_{ab}x_a x_b = 1, \tag{57}$$

$$d_{abc}x_a x_b = (n-1)\left(\frac{N}{n+1} + \frac{1}{2}\right)\Lambda_N x_c. \tag{58}$$

For large N, this reduces to (16). Again, $\mathbb{C}P^n_N$ admits an approximate "south pole" which corresponds to the lowest weight state of V_N. The symmetry group $SU(n+1)$ acts by inner derivation as for $\mathbb{C}P^2$, and the integral is given by the suitably normalized trace over V_N.

3.3. Representation on a Fock space

In order to introduce nontrivial line bundles, it is useful to quantize directly the fibration $U(1) \to S^{2n+1} \to \mathbb{C}P^n$. In this section we will first introduce noncommutative $\mathbb{C}^{n+1}$, in terms of operators a^i, a_i^+ $(i = 1, \ldots, n+1)$ which are quantizations of the coordinate functions $z^i, \overline{z}_i$ of $\mathbb{C}^{n+1} \supset S^{2n+1}$. Then fuzzy $\mathbb{C}P^n$ will be obtained as a subalgebra of this noncommutative $\mathbb{C}^{n+1}$. This has been used first in [22]. The equivalence to the definition given previously will be manifest.

The generators a_i^+, a^i of noncommutative $\mathbb{C}^{n+1}$ are creation- resp. annihilation operators which transform as $V_{(1,0,\ldots,0)}$ resp. $V^*_{(1,0,\ldots,0)} = V_{(0,\ldots,0,1)}$, and satisfy the canonical commutation relations

$$[a^i, a^j] = [a_i^+, a_j^+] = 0, \quad [a^i, a_j^+] = \delta^i_j. \tag{59}$$

The resulting algebra will be denoted as $\mathbb{C}^{n+1}_\theta$.

As in the previous section, we consider the $U(1)$ defined by

$$\omega \circ (a^i, a_i^\dagger) = (a^i\omega, a_i^+\overline{\omega}), \tag{60}$$

where $\omega \in \mathbb{C}$ and $|\omega| = 1$. With this $U(1)$ action, the equivariant operators $C(\kappa, \mathbb{C}^{n+1}_\theta)$ are defined by

$$C(\kappa, \mathbb{C}^{n+1}_\theta) = \{f | f \in \mathrm{Pol}(a^i, a_i^+), i = 1, \ldots, n+1 \quad \text{and} \quad \omega \circ f = f\omega^\kappa\}. \tag{61}$$

This means that κ counts the difference of the number of creation and annihilation operators, and thus an element $f \in C(\kappa, \mathbb{C}_\theta^{n+1})$ satisfies

$$[\mathbf{N}, f] = \kappa f, \tag{62}$$

where $\mathbf{N}$ is the number operator: $\mathbf{N} = \sum_{i=1}^{n+1} a_i^+ a^i$.

As in the commutative case, there is a natural multiplication of the equivariant operators $C(\kappa, \mathbb{C}_\theta^{n+1})$ and $C(\kappa', \mathbb{C}_\theta^{n+1})$ such that

$$C(\kappa, \mathbb{C}_\theta^{n+1}) \times C(\kappa', \mathbb{C}_\theta^{n+1}) \longrightarrow C(\kappa + \kappa', \mathbb{C}_\theta^{n+1}). \tag{63}$$

Hence the equivariant operators $C(\kappa, \mathbb{C}_\theta^{n+1})$ can be interpreted as $C(0, \mathbb{C}_\theta^{n+1})$-module.

Consider the following generators of $C(0, \mathbb{C}_\theta^{n+1})$:

$$\tilde{\xi}_a = a_i^+ (T_a)^i{}_j a^j, \qquad T_a = \tfrac{1}{2}\lambda_a. \tag{64}$$

By construction, they transform in the adjoint of $su(n+1)$, and satisfy the relations

$$[\tilde{\xi}_a, \tilde{\xi}_b] = i f_{abc} \tilde{\xi}_c \tag{65}$$

and

$$\sum_{a=1,\ldots,n^2+2n} \tilde{\xi}_a \tilde{\xi}_a = \frac{n}{2(n+1)} \mathbf{N}(\mathbf{N} + n + 1). \tag{66}$$

To obtain fuzzy $\mathbb{C}P_N^n$, we must “fix the radius” in $\mathbb{C}_\theta^{n+1}$, i.e. choose a Fock space representation with fixed particle number N. As usual, the Fock space is defined by acting with the creation operators a_i^+ on the vacuum state $|0\rangle$, which satisfies $a_i|0\rangle = 0$. The N-particle subspace $\mathcal{F}_N$ is obtained by acting with N creation operators on the vacuum, with basis

$$|\vec{m}\rangle = \sqrt{\frac{m_1! m_2! \cdots m_{n+1}!}{N!}} (a_1^+)^{m_1} (a_2^+)^{m_2} \cdots (a_{n+1}^+)^{m_{n+1}} |0\rangle, \tag{67}$$

where the label $\vec{m}$ is a set of positive integers m_i, $\vec{m} = (m_1, \ldots, m_{n+1})$ satisfying $\sum_i m_i = N$.

Hence if acting on the N-particle subspace $\mathcal{F}_N$, we recover precisely the relations (52) and (53). Next, we verify that the operators $\tilde{\xi}_a$ also satisfy (54) if acting on $\mathcal{F}_N$. In order to see this, we prove the characteristic equation (50) as follows: using the Fierz identity for the generators T^a and the definition of the generators $\tilde{\xi}$, we obtain

$$a_j^+ a^i = 2 \sum T^{ai}{}_j \tilde{\xi}^a + \frac{1}{n+1} \delta^i_j \mathbf{N} = a^i a_j^+ - \delta^i_j. \tag{68}$$

On the other hand, it is straightforward to confirm that the operator P:

$$P^i_j = \frac{1}{\mathbf{N} + 1} a^i a_j^+ \tag{69}$$

is a projection operator, i.e. $P^2 = P$. Defining the matrix $\tilde{X}^i_j = 2 \sum T^{a^i_j} \tilde{\xi}^a$ and using (68), the operator P can be written as

$$P = \frac{\tilde{X} + 1 + \frac{\mathbf{N}}{n+1}}{\mathbf{N} + 1}. \tag{70}$$

In terms of $\tilde{X}$, the relation $P^2 = P$ gives

$$0 = P(P-1) = \frac{1}{(\mathbf{N}+1)^2}\left(\tilde{X} + 1 + \frac{\mathbf{N}}{n+1}\right)\left(\tilde{X} - \mathbf{N} + \frac{\mathbf{N}}{n+1}\right). \tag{71}$$

This is exactly the characteristic equation (50) for X, if the number operator $\mathbf{N}$ is replaced by the number N. Hence the relations (52)–(54) of fuzzy $\mathbb{C}P^n$ hold on the Fock space representation $\mathcal{F}_N$ of $C(0, \mathbb{C}_\theta^{n+1})$.

Therefore any $f \in C(0, \mathbb{C}_\theta^{n+1})$ defines a map

$$f : \mathcal{F}_N \longrightarrow \mathcal{F}_N. \tag{72}$$

Now observe that the Fock space $\mathcal{F}_N$ is precisely the irreducible representation $V_N = V_{(N,0,\ldots,0)}$ of $su(n+1)$. Since the generators $\tilde{\xi}_a$ act on $\mathcal{F}_N$ and generate the $su(n+1)$ algebra, they generate $End_{\mathbb{C}}(\mathcal{F}_N)$, and the equivalence of this definition of $\mathbb{C}P_N^n$ with the one given in the previous section in terms of $Mat(D_N, \mathbb{C})$ (or more precisely $End_{\mathbb{C}}(V_N)$) is manifest[4].

4. Projective modules

From the noncommutative geometry point of view, the algebraic object corresponding to a vector bundle is the projective module of finite type. This equivalence of vector bundles and projective modules is based on the Serre-Swan Theorem, and thus in the following discussion, the projective modules are the relevant objects to deal with.

A projective $\mathcal{A}$-module can be constructed from the free module $\mathcal{A}^p$ together with a projection operator $\mathcal{P}$, which is an element of $Mat(p, \mathcal{A})$, the space of $p \times p$ matrices with elements in the base algebra $\mathcal{A}$.

We will consider the noncommutative analogue of the monopole bundles, i.e. the $U(1)$ bundles over the fuzzy $\mathbb{C}P^n$. For this purpose, we construct a rank 1 projection operator which determines the module associated with the complex rank 1 vector bundle. The advantage of this formulation compared to [4] is that it also provides a canonical connection, which can be used e.g. to calculate Chern numbers.

We follow the approach taken in [26] here. In general, a rank 1 projection operator $\mathcal{P} \in \mathrm{End}\,(\mathcal{A}^p)$ in the space of $p \times p$ matrices can be constructed by using an p-component vector defined as follows:

$$\boldsymbol{v} = (v_\mu), \quad \mu = 1, \ldots, p, \tag{73}$$

where v_μ is an element of a left-$\mathcal{B}$-right-$\mathcal{A}$ bimodule $\mathcal{M}$. Here $\mathcal{B}$ is also an algebra, but not necessarily equivalent to $\mathcal{A}$. The only condition needed is the normalization condition

$$\boldsymbol{v}^\dagger \boldsymbol{v} = \sum_\mu v_\mu^\dagger v_\mu = \mathbf{1}_{\mathcal{B}}, \tag{74}$$

[4] The conjugated version $\mathbb{C}P^{2*}$ would be obtained using the conjugated λ_a matrices $\tilde{\lambda}_a$, and b^{+i}, b_j transforming in the dual representations.

where $\mathbf{1}_{\mathcal{B}}$ denotes the identity of the algebra $\mathcal{B}$. Using this vector $\boldsymbol{v}$, we define the projection operator as

$$\mathcal{P} = \boldsymbol{v}\boldsymbol{v}^{\dagger}. \tag{75}$$

It is apparent that

$$\mathcal{P}\mathcal{P} = \boldsymbol{v}(\boldsymbol{v}^{\dagger}\boldsymbol{v})\boldsymbol{v}^{\dagger} = \boldsymbol{v}\mathbf{1}_{\mathcal{B}}\boldsymbol{v}^{\dagger} = \mathcal{P} \tag{76}$$

by the normalization condition.

Note that in order to define the projective module, the matrix elements of the projection operator $\mathcal{P}$ must be elements of the algebra $\mathcal{A}$. However, this does not mean that each element of the vector $\boldsymbol{v}$ is also an element of $\mathcal{A}$. Recall that a similar situation occurs in the construction of the so-called localized instanton in $\mathbb{R}^4_\theta$ using the ADHM construction [28,29,1].

When we construct the vector $\boldsymbol{v}$ below, we take the elements to be in $C(\kappa, \mathbb{C}^{n+1}_\theta)$. Then the matrix elements of the projection operator $\mathcal{P}_\kappa$ defined as in (75) are indeed elements of $C(0, \mathbb{C}^{n+1}_\theta)$, which if acting on $\mathcal{F}_N$ is just $\mathbb{C}P^n_N$, as it should.

To define the vector $\boldsymbol{v}$, we should distinguish two cases depending on the sign of the integer κ:

1. For $0 < \kappa$:
$$v(\vec{j}) = (a^1)^{j_1}(a^2)^{j_2}\cdots(a^{n+1})^{j_{n+1}}c_+(\vec{j}), \tag{77}$$
where the dimension of the vector $\boldsymbol{v}$ is $D_\kappa = \frac{(n+\kappa)!}{n!\kappa!}$, $\vec{j} = (j_1, \ldots, j_n)$ where $j_i \geq 0$ are integers with $\sum j_i = \kappa$. The normalization factor is
$$(c_+(\vec{j}))^2 = \frac{\kappa!}{j_1!j_2!\cdots j_{n+1}!\mathbf{N}(\mathbf{N}-1)\cdots(\mathbf{N}-\kappa+1)}. \tag{78}$$
2. For $-N < \kappa < 0$:
$$v_\mu = (a_1^+)^{j_1}(a_2^+)^{j_2}\cdots(a_{n+1}^+)^{j_{n+1}}c_-(\vec{j}), \tag{79}$$
where the dimension is $D_\kappa = \frac{(n+|\kappa|)!}{n!|\kappa|!}$, and the normalization is
$$(c_-(\vec{j}))^2 = \frac{|\kappa|!}{j_1!j_2!\cdots j_{n+1}!(\mathbf{N}+n+1)(\mathbf{N}+n+2)\cdots(\mathbf{N}+n+|\kappa|)}. \tag{80}$$

It is easy to verify in both cases that $\boldsymbol{v}^{\dagger}\boldsymbol{v} = \mathbf{1}_{\mathcal{B}}$.

One might worry that the denominator of the normalization factor $c_+(\vec{j})$ can become 0 for large $\kappa > 0$. However, when constructing the projection operator $\mathcal{P}_\kappa = \boldsymbol{v}\boldsymbol{v}^{\dagger}$ and specifying the representation space to be $\mathcal{F}_N$, the number operator $\mathbf{N}$ in c_+ is replaced by the value $N + \kappa$. Therefore for $\kappa > 0$ all expressions are well-defined.

On the other hand, for $\kappa < 0$, there is a limit for the admissible values of κ. The reason is that $\boldsymbol{v}^{\dagger}$ contains $|\kappa|$ annihilation operators, hence

$$\mathcal{P}_\kappa = \boldsymbol{v}\boldsymbol{v}^{\dagger}$$

acting on $\mathcal{F}_N$ is ill-defined if $|\kappa| > N$. Therefore we must impose the bound $\kappa + N \geq 0$ in the case $\kappa < 0$.

4.1. Scalar fields, or sections in line bundles

Now we can consider the projective module $\Gamma_\kappa(\mathbb{C}P^n_N) = \mathcal{P}_\kappa(\mathbb{C}P^n_N)^{D_{|\kappa|}}$. An element ξ of the module $\Gamma_\kappa(\mathbb{C}P^n_N)$ is thus a $D_{|\kappa|}$-dimensional vector $\xi = \{\xi_\mu\}$, the components of which are $\xi_\mu \in \mathbb{C}P^n_N$. This is a section of the line bundle, corresponding to a complex scalar field in $U(1)$ gauge theory. However, the complex scalar field in $U(1)$ gauge theory has a single component in the conventional field theory formulation. The relation between these formulations will be explained next.

Assume that $\kappa + N \geq 0$. The single-component scalar field, i.e. section of the monopole bundle, is given by

$$\hat{\xi} = \boldsymbol{v}^\dagger \xi = \sum_\mu v^\dagger_\mu \xi_\mu. \tag{81}$$

On the other hand, we can act with $\hat{\xi}$ on an element $\psi = \sum_m f_m|m\rangle \in \mathcal{F}_N$, with the result

$$\begin{aligned}\hat{\xi}\psi &= \left(\sum_\mu v^\dagger_\mu \xi_\mu\right)\left(\sum_m f_m|m\rangle\right) \\ &= \sum_{\substack{p_i\in\mathbb{Z}_+,\\ p_1+\cdots+p_{n+1}=N+\kappa}} f'_{p_1\cdots p_{n+1}}(a^+_1)^{p_1}\cdots(a^+_{n+1})^{p_{n+1}}|0\rangle,\end{aligned} \tag{82}$$

i.e.

$$\hat{\xi}\psi \in \mathcal{F}_{N+\kappa}. \tag{83}$$

Thus we can identify an element $\hat{\xi} \in C(\kappa, \mathbb{C}P^n_N)$ with a map

$$\hat{\xi} \in Hom_{\mathbb{C}}(\mathcal{F}_N, \mathcal{F}_{N+\kappa}) : \mathcal{F}_N \longrightarrow \mathcal{F}_{N+\kappa} \tag{84}$$

in agreement with [4]. In other words, we can identify the scalar field $\hat{\xi}$ on $\mathbb{C}P^n_N$ with monopole charge κ with a $D_N \times D_{(N+\kappa)}$ rectangular matrix. Equivalently, we can identify the section of the line bundle over $\mathbb{C}P^n_N$ given by $\hat{\xi} \in C(\kappa, \mathbb{C}^{n+1}_\theta)_N$ as $\mathbb{C}P^n_{N+\kappa}$–$\mathbb{C}P^n_N$ bimodule.

Note that from this construction, it is apparent that we must impose the bound $\kappa + N \geq 0$.

From the above construction, we see that there are two pictures of the monopole bundle $\mathbb{C}P^n_{N+\kappa}$–$\mathbb{C}P^n_N$ bimodule. Namely, the same bimodule can be obtained from fuzzy $\mathbb{C}P^n_{N+\kappa}$ with monopole charge $-\kappa$ (assuming $N + \kappa > 0$). Since in noncommutative algebras we have to make a choice of left and right multiplication on the module, we can find two equivalent bimodules as

1. Monopole with charge $|\kappa|$ on $\mathbb{C}P^n_N$.
2. Monopole with charge $-|\kappa|$ on $\mathbb{C}P^n_{N+|\kappa|}$.

Hence there is a duality between $\mathbb{C}P^n_N$ with monopole charge κ and $\mathbb{C}P^n_{N+\kappa}$ with monopole charge $-\kappa$. We see that $\mathbb{C}P^n_N$ and $\mathbb{C}P^n_{N+|\kappa|}$ are Morita equivalent, and the scalar field is the equivalence bimodule (the inverse of $\hat{\xi}$ is given by its conjugate). This is an example showing the relation of Morita equivalence and duality of the noncommutative space.

5. Differential calculus, connection and field strength

5.1. Differential forms

We introduce a basis of one-forms θ_a, $a = 1, 2, \ldots, n^2 + 2n$ à la Madore [2], which transform in the adjoint of $su(n+1)$ and commute with the algebra of functions:

$$[\theta_a, f] = 0, \qquad \theta_a\theta_b = -\theta_b\theta_a. \tag{85}$$

This defines a space of exterior forms on fuzzy $\mathbb{C}P^n_N$, which we denote by $\Omega^*_N := \Omega^*(\mathbb{C}P^n_N)$. The gradation given by the number of anticommuting generators θ_a. The highest non-vanishing form is the $(n^2 + 2n)$-form corresponding to the volume form of $su(n+1)$.

One can also define an exterior derivative $d : \Omega^k_N \to \Omega^{k+1}_N$ such that $d^2 = 0$ and imposing the graded Leibniz rule. Its action on the algebra elements $f \in \Omega^0_N$ is given by the commutator with a special one-form: Consider the invariant one-form

$$\Theta = \xi_a\theta_a. \tag{86}$$

Then the exterior derivative of a function $f \in \mathbb{C}P^n_N$ is given by

$$\mathrm{d}f := [\Theta, f] = [\xi_a, f]\theta_a. \tag{87}$$

In particular, we have

$$\mathrm{d}\xi_b = [\xi_a, \xi_b]\theta_a = if_{abc}\theta_a\xi_c. \tag{88}$$

The definition of d on higher forms is straightforward, once we find $d : \Omega^1_N \to \Omega^2_N$ such that $d^2(f) = 0$. To find it, we follow the approach of [30] for the q-fuzzy sphere. Notice first that there is a natural bimodule-map from one-forms to 2-forms, given by

$$\star_1(\theta_a) := \frac{i}{2} f_{abc}\theta_b\theta_c. \tag{89}$$

Then we define

$$d : \ \Omega^1_N \to \Omega^2_N, \qquad \alpha \ \mapsto \mathrm{d}\alpha = [\Theta, \alpha]_+ - \star_1(\alpha), \tag{90}$$

where $\alpha \in \Omega^1_N$. One can verify $d^2 = 0$ in general. To see this, note that

$$\mathrm{d}f = [\Theta, \mathrm{d}f]_+ - \star_1(\mathrm{d}f) = 0$$

using the following relation:

$$\star_1(\Theta) = \Theta^2. \tag{91}$$

This follows from

$$\Theta^2 = \Theta\Theta = \frac{1}{2}\theta_a\theta_b[\xi_a, \xi_b] = i\frac{1}{2} f_{abc}\theta_a\theta_b\xi_c =: \frac{i}{\Lambda_N}\eta, \tag{92}$$

where $\eta = \frac{1}{2}\theta_a\theta_b f_{abc}x_c$ is the symplectic form. One can also show that

$$\mathrm{d}\Theta = \Theta^2, \tag{93}$$

which implies

$$\mathrm{d}\eta = 0. \tag{94}$$

Furthermore, using group-theoretic arguments[5] it is easy to see that

$$\eta = C\frac{1}{2} f_{abc}\, \mathrm{d}x_a\, \mathrm{d}x_b x_c \tag{95}$$

for some numerical constant C. This is more easily recognized as symplectic form.

In order to extend this calculus to $\mathbb{C}_\theta^{n+1}$, we can express the generator Θ in terms of the generators a^j, a_i^+ as above, interpreted as quantizations of the coordinate functions z^j, $\overline{z}_i$ on $\mathbb{C}^{n+1}$:

$$\Theta = a_i^+ T^i_{a\,j} a^j \theta_a. \tag{96}$$

Then the calculus on $\mathbb{C}P_N^n$ naturally induces a calculus on $\mathbb{C}_\theta^{n+1}$.

5.1.1. Relation to the classical case

For later use, we want to calculate the constant C in (95) in the classical limit $N \to \infty$. Consider $\mathbb{C}P^2$ for simplicity. We introduce a normalization of the frame by

$$\langle\theta_a, \theta_b\rangle = c\delta_{ab}, \tag{97}$$

where the constant c is determined such that the tangential one-forms are properly normalized: using (88), we have

$$\langle \mathrm{d}y_a, \mathrm{d}y_b\rangle = -f_{ras} f_{ubv} y_s y_v \langle\theta_r, \theta_u\rangle = -c f_{ars} f_{brv} y_s y_v \tag{98}$$

(recall that y_a denotes the classical coordinate functions). It is sufficient to consider the "south pole" of $\mathbb{C}P^2$, where $y_a = -\delta_{a,8}$ as discussed in Section 2.2 (setting $r = 1$). Then

$$\langle \mathrm{d}y_a, \mathrm{d}y_b\rangle = -c f_{ar8} f_{br8} \overset{!}{=} \delta_{ab}^{(\mathrm{tang})}, \tag{99}$$

which due to the explicit form of f_{ab8} is non-vanishing only for tangential $\mathrm{d}y_a$, i.e. $a, b \in \{4, 5, 6, 7\}$ as in (20). This shows that the "non-tangential" one-forms on $\mathbb{C}P^2$ have zero norm, and indeed the correct four-dimensional calculus on $\mathbb{C}P^2$ is recovered from our construction. In the fuzzy case, the additional one-forms cannot be avoided, however.

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[5] Using (88) we can write $f_{abc}\, \mathrm{d}x_a\, \mathrm{d}x_b x_c \propto \sum xxx\theta\theta$, which must be a singlet. Now $xxx \in V_{(3,3)} \oplus V_{(2,2)} \oplus V_{(1,1)}$, but only $V_{(1,1)}$ can be contracted with $\theta\theta$ to give a singlet.

The constant c can now be calculated by summing over the tangential $a, b \in \{4, 5, 6, 7\}$, which gives

$$4 = \delta_{ab}^{(\mathrm{tang})} \langle \mathrm{d}y_a, \mathrm{d}y_b \rangle = \delta_{ab} \langle \mathrm{d}y_a, \mathrm{d}y_b \rangle = -c f_{ras} f_{rav} y_s y_v = -3c. \tag{100}$$

Here we used the result that $\langle \mathrm{d}y_a, \mathrm{d}y_b \rangle = 0$ for non-tangential forms, and

$$f_{ras} f_{rav} = 3\delta_{sv} \tag{101}$$

for $su(3)$. Therefore

$$c = -\tfrac{4}{3}. \tag{102}$$

We can now relate $\eta = \frac{1}{2}\theta_a \theta_b f_{abc} y_c$ to

$$\eta' = \tfrac{1}{2} f_{abc} dy_a dy_b y_c. \tag{103}$$

Comparing $\langle \eta, \eta \rangle$ with $\langle \eta', \eta' \rangle$, we obtain

$$\eta' = \frac{1}{2c} f_{abc} \theta_a \theta_b y_c = \frac{1}{c}\eta, \tag{104}$$

therefore

$$\eta' = -\tfrac{3}{4}\eta \tag{105}$$

for $\mathbb{C}P^2$. For $\mathbb{C}P^n$, the same calculation gives

$$\eta' = -\frac{n+1}{2n}\eta. \tag{106}$$

5.2. Canonical connection and field strength

Once we have a differential calculus, we can define the canonical connection ∇ over the projective module $\Gamma_\kappa(\mathbb{C}P^n_N)$ defined by the projection $\mathcal{P}_\kappa$ by

$$\mathcal{P}_\kappa \, \mathrm{d}\xi, \tag{107}$$

where $\xi \in \Gamma_\kappa(\mathbb{C}P^n_N)$. The curvature 2-form of this canonical connection is given by $\mathcal{P}_\kappa \, \mathrm{d}\mathcal{P}_\kappa \, \mathrm{d}\mathcal{P}_\kappa$.

When the connection is represented by the covariant derivative on the scalar field $\hat{\xi}$, we obtain

$$\nabla\hat{\xi} \equiv \boldsymbol{v}^\dagger(\mathcal{P}_\kappa \, \mathrm{d}\xi) = (\mathrm{d} + \boldsymbol{v}^\dagger \, \mathrm{d}\boldsymbol{v})\hat{\xi}. \tag{108}$$

The gauge field and field strength of the above connection ∇ is given by

$$A = \boldsymbol{v}^\dagger \mathrm{d}\boldsymbol{v} = \boldsymbol{v}^\dagger \Theta \boldsymbol{v} - \Theta, \tag{109}$$

$$F = \boldsymbol{v}^\dagger \mathcal{P}_\kappa \, \mathrm{d}\mathcal{P}_\kappa \, \mathrm{d}\mathcal{P}_\kappa \boldsymbol{v} = \boldsymbol{v}^\dagger \Theta \boldsymbol{v}\boldsymbol{v}^\dagger \Theta \boldsymbol{v} - \boldsymbol{v}^\dagger \Theta^2 \boldsymbol{v}. \tag{110}$$

In order to evaluate this expression, we extend the differential calculus from $\mathbb{C}P^n_N$ to $\mathbb{C}^{n+1}_\theta$ as discussed above, postulating that the θ_a commute with all a^i, a_j (this can also be interpreted as a calculus on the $U(1)$ principal bundle over $\mathbb{C}P^n_N$).

Assume first $\kappa > 0$. Then using

$$a^j \mathbf{N} = (\mathbf{N}+1)a^j, \qquad \mathbf{N}a_i^+ = a_i^+(\mathbf{N}+1), \tag{111}$$

we have

$$\begin{aligned} \boldsymbol{v}^\dagger \Theta \boldsymbol{v} &= \boldsymbol{v}^\dagger a_i^+ a^j \boldsymbol{v}\, T^i_{a\,j}\theta_a = a_i^+ \boldsymbol{v}^\dagger(\mathbf{N}+1)\boldsymbol{v}(\mathbf{N}+1)a^j T^i_{a\,j}\theta_a \\ &= \frac{N-\kappa}{N} a_i^+ a^j T^i_{a\,j}\theta_a = \frac{N-\kappa}{N}\Theta \end{aligned} \tag{112}$$

since

$$\boldsymbol{v}^\dagger(\mathbf{N}+1)\boldsymbol{v}(\mathbf{N}+1) = \frac{\mathbf{N}+1-\kappa}{\mathbf{N}+1} \tag{113}$$

by construction.

Similarly:

$$\begin{aligned} \boldsymbol{v}^\dagger \Theta^2 \boldsymbol{v} &= \boldsymbol{v}^\dagger (a_i^+ T^i_{a\,j} a^j\, \theta_a)(a_l^+ T^l_{bk} a^k \theta_b)\boldsymbol{v} \\ &= \boldsymbol{v}^\dagger a_i^+ T^i_{a\,j} T^j_{b\,k} a^k \boldsymbol{v}\theta_a\theta_b + \boldsymbol{v}^\dagger a_i^+ a_l^+ a^j a^k \boldsymbol{v} T^i_{a\,j} T^l_{bk}\theta_a\theta_b, \end{aligned} \tag{114}$$

which, using

$$T_a T_b = \frac{1}{2(n+1)}\delta_{ab} + \frac{1}{2}(if_{abc} + d_{abc})T_c \tag{115}$$

is

$$\begin{aligned} \boldsymbol{v}^\dagger \Theta^2 \boldsymbol{v} &= \boldsymbol{v}^\dagger a_I^+ \left(\frac{1}{2(n+1)}\delta_{ab}\delta^i_k + \frac{1}{2}(i f_{abc} + d_{abc})T^i_{c\,k} \right) a^k \boldsymbol{v}\theta_a\theta_b \\ &\quad + a_i^+ a_l^+ \boldsymbol{v}^\dagger(\mathbf{N}+2)\boldsymbol{v}(\mathbf{N}+2)a^j a^k T^i_{a\,j} T^l_{bk}\theta_a\theta_b \\ &= \frac{i}{2}\boldsymbol{v}^\dagger a_i^+ f_{abc} T^i_{c\,k} a^k\, \boldsymbol{v}\theta_a\theta_b + \frac{N-\kappa}{N} a_i^+([a_l^+, a^j] + a^j a_l^+)a^k T^i_{a\,j} T^l_{bk}\theta_a\theta_b \\ &= \frac{i}{2} a_i^+ \boldsymbol{v}^\dagger(\mathbf{N}+1)\boldsymbol{v}(\mathbf{N}+1) f_{abc} T^i_{c\,k} a^k \theta_a\theta_b - \frac{N-\kappa}{N} a_i^+ a^k T^i_{a\,j} T^j_{b\,k}\theta_a\theta_b \\ &\quad + \frac{N-\kappa}{N} a_i^+ T^i_{a\,j} a^j a_l^+ T^l_{bk} a^k \theta_a\theta_b = \frac{N-\kappa}{N}\Theta^2. \end{aligned} \tag{116}$$

Therefore

$$F = \left(\left(\frac{\mathbf{N}-\kappa}{\mathbf{N}}\right)^2 - \left(\frac{\mathbf{N}-\kappa}{\mathbf{N}}\right) \right)\Theta^2 = \frac{-\kappa(\mathbf{N}-\kappa)}{\mathbf{N}^2 \Lambda_N} i\eta = \frac{-\kappa N}{(N+\kappa)^2 \Lambda_N} i\eta, \tag{117}$$

where we have used that when the field strength F is evaluated over the scalar field $\hat{\xi}$, the number operator takes the value $\mathbf{N} = N + \kappa$. Similarly for $\kappa < 0$, we have

$$F = \frac{|\kappa|(\mathbf{N}+n+|\kappa|+1)}{(\mathbf{N}+n+1)^2 \Lambda_N} i\eta = \frac{|\kappa|(N+n+1)}{(N-|\kappa|+n+1)^2 \Lambda_N} i\eta. \tag{118}$$

Thus in the large N limit, we obtain for all κ

$$F = -\kappa\sqrt{\frac{n}{2(n+1)}}i\eta = \kappa\sqrt{2}\left(\frac{n}{n+1}\right)^{3/2} i\eta' \tag{119}$$

using (106), up to $+o(1/N)$ corrections. Hence the field strength is indeed quantized, and it is a multiple of the symplectic form in the large N limit. In the next section, we verify that the first Chern number c_1 is given by $-\kappa$ in the classical limit.

5.3. Calculation of the first Chern number for $N \to \infty$

In the classical case, we can integrate the symplectic form η' over the cycle $y_1^2 + y_2^2 + y_3^2 = \frac{n+1}{2n}$ in $\mathbb{C}P^n$ found in Section 2.2. Using $f_{abc} = \varepsilon_{abc}$ for $a, b, c \in \{1, 2, 3\}$, we have

$$\int_{S_R^2} \eta' = \int_{S_R^2} \frac{1}{2}\varepsilon_{abc} y_a \,\mathrm{d}y_b \,\mathrm{d}y_c = 4\pi R^3, \tag{120}$$

where the sphere has radius $R^2 = \frac{n+1}{2n}$. This shows in particular that these spheres are indeed non-trivial. Therefore using (119), the first Chern number is

$$c_1 = \frac{i}{2\pi}\int_{S_R^2} F = -\kappa \tag{121}$$

in the commutative limit $N \to \infty$. This shows that the bundles constructed above should be interpreted as noncommutative versions of the classical monopole bundles with Chern number $c_1 = -\kappa$.

5.4. Discussion on Chern numbers for finite N

In the fuzzy case (i.e. for finite N), it is very difficult to give a satisfactory definition of Chern numbers. One reason for this is the lack of a differential calculus with the appropriate dimensions for finite N. However, it is known e.g. from recent investigations of fuzzy spheres [14] that it is still possible to write down suitable integrals in the fuzzy case, which in the large N (i.e. the commutative) limit reproduce the usual Chern numbers, but which are neither topological nor integer for finite N. We will refer to such prescriptions as “asymptotic” Chern numbers. They are still useful since they produce numbers in the fuzzy case which reduce to the usual (integer) Chern numbers in the classical limit. We illustrate this for fuzzy $\mathbb{C}P_N^n$ by giving a prescription to calculate such an “asymptotic” Chern number c_1, integrating $\frac{i}{2\pi}F$ over a suitable “fuzzy sub-sphere”.

5.4.1. Fuzzy sub-spheres

In the classical case, c_1 can be obtained by integrating $\frac{i}{2\pi}F$ over any 2-sphere in $\mathbb{C}P^n$. Since $\mathbb{C}P_N^n$ is defined in terms of a simple matrix algebra, it does not admit any non-trivial subspaces $\mathbb{C}P_N^n/\mathcal{I}$ defined by some two-sided ideal $\mathcal{I}$. Therefore in order to compute the first “asymptotic” Chern number, we have to relax the concept of a subspace in the fuzzy case. A natural way to do this in our context is the following.

Note that for any given root α of $su(n+1)$, $V_N \cong \mathcal{F}_N$ decomposes into a direct sum of irreps of $su(2)_\alpha \subset su(n+1)$. Now fix a root α. Then there is precisely one such irrep denoted by $H^{\alpha,N}$ which has maximal dimension $N+1$ (note that the weights of V_N form a simplex in weight lattice of size $N+1$). The other irreps have smaller dimension $M+1 \leq N+1$, denoted by $H^{\alpha,M}$ (we omit additional labels for simplicity). Let $P^{\alpha,M} : V_N \to H^{\alpha,M}$ be the projector on $H^{\alpha,M}$. Now we can define maps

$$\mathbb{C}P^n_N \cong End(V_N) \to End(H^{\alpha,M}), \qquad f \mapsto \hat{f} := P^{\alpha,M} f P^{\alpha,M}. \tag{122}$$

In principle, each

$$S^2_{\alpha,M} := End(H^{\alpha,M}) \tag{123}$$

could be considered as a fuzzy sphere, but not necessarily as sub-spheres of $\mathbb{C}P^n_N$. However, we shall explain below that the maximal $S^2_{\alpha,N}$ can be considered as an "asymptotic sub-sphere" of $\mathbb{C}P^n_N$, in the sense that it becomes the algebra of functions on a sub-sphere of $\mathbb{C}P^n$ in the large N limit. This sub-sphere in fact coincides with the non-trivial 2-cycles found in Section 2.2 for suitable choice of the root α. To see this, consider the corresponding projected coordinate generators

$$\hat{x}_a = P^{\alpha,M} x_a P^{\alpha,M} \tag{124}$$

obtained from the fuzzy coordinate functions of $\mathbb{C}P^n_N$. If $S^2_{\alpha,M}$ is to be interpreted as sub-sphere of $\mathbb{C}P^n_N$, then $\hat{x}_a$ should be interpreted as restriction (or pull-back) of the coordinate function x_a of $\mathbb{C}P^n_N$ to $S^2_{\alpha,M}$. However, it is easy to see that[6]

$$g_{ab}\hat{x}_a\hat{x}_b < 1 \tag{125}$$

for finite N, which is in contrast to the constraint $g_{ab}x_a x_b = 1$ (57) of $\mathbb{C}P^n_N$. The reason is that the (rescaled) quadratic Casimir of $su(n+1)$, which can be written as

$$g_{ab}x_a x_b = \sum_\beta \left(\frac{1}{2}x^+_\beta x^-_\beta + \frac{1}{2}x^-_\beta x^+_\beta\right) + \sum_i H_i^2 = 1, \tag{126}$$

where $\sum_\beta$ goes over all positive roots of $su(n+1)$, and H_i are the (suitably rescaled) Cartan generators. Now the restricted generators are related to the unrestricted ones as follows:

$$\hat{x}^\pm_\alpha = x^\pm_\alpha, \qquad \hat{H}_i = H_i, \qquad \hat{x}^\pm_\beta = 0 \quad \text{for } \beta \neq \alpha, \tag{127}$$

because $x^\pm_\beta$ does not preserve any $H^{\alpha,M}$ for $\beta \neq \alpha$. This implies (125), since $(\frac{1}{2}x^+_\beta x^-_\beta + \frac{1}{2}x^-_\beta x^+_\beta)$ is positive definite. This reflects the fact that $\mathbb{C}P^n_N$ does not admit any (strict) sub-spaces. However, we can consider "asymptotic sub-spaces" by relaxing the constraint $g_{ab}x_a x_b = 1$ and allow for "quantum corrections" of order $1/N$, requiring only

$$g_{ab}\hat{x}_a\hat{x}_b = 1 - \mathrm{O}\frac{1}{N}. \tag{128}$$

This holds indeed for $H^{\alpha,N}$, but in general not for $H^{\alpha,M}$ with $M < N$. To see this, recall that the rising- and lowering operators act as $\xi^+_\beta v_k = \sqrt{(M-k)k}\, v_{k+1}$ where $\{v_k\}$ is the

[6] This inequality is to be understood in the operator-norm.

normalized weight basis of the $su(2)_\beta$ irrep $H^{\beta,M}$, and similar for ξ^-_β. Including the scale factor $Ł_N = O(1/N)$, this implies

$$x^\pm_\beta = \mathrm{O}\frac{1}{\sqrt{N}}, \tag{129}$$

if acting on $H^{\alpha,N}$ (recall that the weights of V_N form a simplex in weight lattice of size N, and $H^{\alpha,N}$ forms an edge of this simplex), while it is not true in the bulk of V_N i.e. on general $H^{\alpha,M}$. Therefore we can consider $S^2_{\alpha,N}$ as "asymptotic subspace" of $\mathbb{C}P^n_N$, which in fact reduces for $N \to \infty$ to the non-trivial 2-cycles found in Section 2.2. This will be used in the calculation of asymptotic first Chern numbers below.

5.4.2. Asymptotic first Chern number

To compute the Chern number c_1, we have to integrate $\frac{i}{2\pi}F$ over a 2-cycle in $\mathbb{C}P^n_N$. According to the above discussion, we give a definition in terms of an integral over a specific fuzzy $S^2_{\alpha,N} \cong \mathbb{C}P^1_N$ which is an asymptotic subspace of $\mathbb{C}P^n_N$ in the above sense, and then perform the integration.

In order to specify the $S^2_{\alpha,N}$, we first choose a Hilbert space $H^{\alpha,N}$ with maximal dimension, denoted by $\mathcal{F}_S$. It can be defined as

$$\mathcal{F}_S = \{|\psi\rangle \in \mathcal{F}_N; a_i|\psi\rangle = 0 \quad \text{for} \quad i = 3, \dots, n+1 \ \text{ and } \ \mathbf{N}|\psi\rangle = N|\psi\rangle\}. \tag{130}$$

The space $\mathcal{F}_S$ has the same dimension as the Hilbert space of fuzzy $\mathbb{C}P^1_N$. The generators associated with the $su(2)$ rotations of this fuzzy $\mathbb{C}P^1$ are

$$T_m = \frac{1}{2}\begin{pmatrix} \sigma^m & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}. \tag{131}$$

The corresponding coordinates are linear combinations of coordinate operators of $\mathbb{C}P^n_N$, and we denote them by $\hat{x}_m = \Lambda_N a^+_i T^m_{ij} a_j$ in agreement with the notation of the previous section. The radius of fuzzy $\mathbb{C}P^1$ is defined by these coordinates $\hat{x}_m$ as

$$\sum_{m=1}^{3} (\hat{x}_m)^2 = \frac{1}{4}\Lambda_N^2 N(N+2) = R_N^2. \tag{132}$$

Representing the algebra generated by $\hat{x}_m$ on $\mathcal{F}_S$, we obtain precisely the matrix algebra of fuzzy $\mathbb{C}P^1_N$. Therefore we can use the standard results of the integration over $\mathbb{C}P^1_N$. We introduce the volume element of this fuzzy $\mathbb{C}P^1_N$,

$$\omega = \frac{1}{2R_N}\epsilon_{mnp}\hat{x}_m\, \mathrm{d}\hat{x}_n\, \mathrm{d}\hat{x}_p \in \Omega^2,$$

which is an invariant 2-form and agrees with the volume element of the sphere with radius $R = \sqrt{\frac{n+1}{2n}}$ in the commutative limit $N \to \infty$. The integration $\int : \Omega^2 \to \mathbb{C}$ over this fuzzy

$\mathbb{C}P^1_N$ is then defined by [14]

$$\frac{1}{2}\int_{\mathbb{C}P^1_N} f\epsilon_{mnp}\hat{x}_m\,\mathrm{d}\hat{x}_n\,\mathrm{d}\hat{x}_p \equiv 4\pi R_N^3\,\mathrm{Tr}_{\mathcal{F}_S}\{f\}, \tag{133}$$

where $\mathrm{Tr}_{\mathcal{F}_S}$ denotes the trace over $\mathcal{F}_S$ normalized such that $\mathrm{Tr}_{\mathcal{F}_S}\{1\} = 1$. Using the commutation relations of $\hat{x}_m$ and the definition of the derivatives defined in the previous section, we obtain

$$\omega = R_N\left(\frac{1}{\mathbf{N}(\mathbf{N}+2)} - \frac{1}{2}\right)\epsilon_{mnp}\hat{x}_m\hat{\theta}_n\hat{\theta}_p, \tag{134}$$

where the $\hat{\theta}_m$, $m = 1,2,3$ is the one form over $\mathbb{C}P^1_N$.

The Chern character c_1 is defined by using the projection operators $\mathcal{P}_\kappa$. To evaluate it we first take the $\mathbb{C}P^n_N$ -valued trace $\mathrm{Tr}_\kappa\{\cdot\}$ over $End(\Gamma_\kappa(\mathbb{C}P^n_N))$, and then integrate over $\mathbb{C}P^1_N$:

$$c_1 = \frac{i}{2\pi}\int_{\mathbb{C}P^1_N}\mathrm{Tr}_\kappa\{\mathcal{P}_\kappa\,\mathrm{d}\mathcal{P}_\kappa\,\mathrm{d}\mathcal{P}_\kappa\} = \frac{i}{2\pi}\int_{\mathbb{C}P^1_N}\sum_\mu v_\mu F_\kappa v_\mu^\dagger, \tag{135}$$

where F_κ is the field strength F given either in Eq. (117) or (118), depending on the value of κ.

Using the definition (77) and (79) of v_μ we can perform the summation over μ and find the following expression for the integrand:

For $\kappa > 0$:

$$\sum_\mu v_\mu F_\kappa v_\mu^\dagger = \frac{(\mathbf{N}+n+\kappa+1)\cdots(\mathbf{N}+n+2)}{(\mathbf{N}+1)\cdots(\mathbf{N}+\kappa)}\frac{-i\kappa\mathbf{N}}{2(\mathbf{N}+\kappa)^2\Lambda_N}f_{abc}x_a\theta_b\theta_c. \tag{136}$$

For $\kappa < 0$:

$$\sum_\mu v_\mu F_\kappa v_\mu^\dagger = \frac{-\mathrm{i}\kappa(\mathbf{N}-|\kappa|)\cdots(\mathbf{N}-1)}{(\mathbf{N}+n-|\kappa|+1)\cdots(\mathbf{N}+n)}\frac{(\mathbf{N}+n+1)}{2(\mathbf{N}-|\kappa|+n+1)^2\Lambda_N}f_{abc}x_a\theta_b\theta_c. \tag{137}$$

These are 2-forms over $\mathbb{C}P^n_N$. In order to integrate them over the fuzzy sub-space $\mathbb{C}P^1_N$, we should also pull-back these 2-forms to $\mathbb{C}P^1_N$. To do this we split the coordinates into the two orthogonal sets, $(x_m, x_{m^\perp})$, where x_m corresponds to the $SU(2)_\alpha$. Correspondingly, we split the one-forms into $(\theta_m, \theta_{m^\perp})$. This means that θ_m is the dual of $\partial_m = \frac{1}{\Lambda_N}ad_{x_m}$, analogously to the commutative case. Now we define the pull-back by projecting out $\theta_{m^\perp}$ and identify $\hat{\theta}_m$ with θ_m.

Since $\mathbb{C}P^1_N$ corresponds to a $su(2)$ subalgebra of $su(n+1)$, this implies that the pull-back of $f_{abc}x_a\theta_b\theta_c$ is $\epsilon_{mnp}\hat{x}_m\hat{\theta}_n\hat{\theta}_p$. This can now be integrated over fuzzy $\mathbb{C}P^1_N$ with radius R_N according to (133), and we get from (135) for $\kappa > 0$

$$\begin{aligned} c_1 &= \frac{i}{2\pi}\frac{(N+n+\kappa+1)\cdots(N+n+2)}{(N+1)\cdots(N+\kappa)}\frac{-\mathrm{i}\kappa N}{2(N+\kappa)^2\Lambda_N}\int_{\mathbb{C}P^1_N}\epsilon_{mnp}(\hat{x}_m\hat{\theta}_n\hat{\theta}_p) \\ &= -\kappa\frac{(N+n+\kappa+1)\cdots(N+n+2)}{(N+1)\cdots(N+\kappa)}\frac{N}{(N+\kappa)^2}\frac{\sqrt{N(N+2)}}{\left(1-\frac{2}{N(N+2)}\right)}. \end{aligned} \tag{138}$$

For large N this yields

$$c_1 = -\kappa + \frac{1}{N}(\kappa(n-1)+1) + \cdots. \tag{139}$$

In the same way we get for $\kappa < 0$:

$$c_1 = -\kappa \frac{(N-|\kappa|)\cdots(N-1)}{(N+n-|\kappa|+1)\cdots(N+n)} \frac{(N+n+1)}{(N-|\kappa|+n+1)^2} \frac{\sqrt{N(N+2)}}{\left(1-\frac{2}{N(N+2)}\right)}. \tag{140}$$

The expansion with respect to $\frac{1}{N}$ is

$$c_1 = -\kappa\left(1 + \frac{1}{N}(-\kappa(n-1)-n) + \cdots\right). \tag{141}$$

6. Conclusion

In this paper, we investigated the definition of fuzzy complex projective space $\mathbb{C}P^n_N$ from two different points of view, and constructed the nontrivial $U(1)$ bundles over those spaces. The corresponding Chern classes are calculated.

The first approach is to consider $\mathbb{C}P^n$ as (co)adjoint orbit, given by $(n+1)\times(n+1)$ matrices Y which satisfy a certain characteristic equation. The quantization of the function algebra is given by a simple matrix algebra $Mat(D_N, \mathbb{C})$, more precisely $End_{\mathbb{C}}(V_N)$ for certain irreducible representations V_N of $su(n+1)$. The appropriate representations V_N are determined using harmonic analysis. This leads to an algebra-valued $(n+1)\times(n+1)$ matrix X, whose characteristic equation gives the explicit relations satisfied by the fuzzy coordinate functions.

The second approach uses the generalized Hopf fibration $U(1) \to S^{2n+1} \to \mathbb{C}P^n$. Again a characteristic equation is derived for a certain operator-valued matrix, which coincides with the first approach when we specify the Fock space representation $\mathcal{F}_N \cong V_N$. The second construction is very useful to define the projective modules.

We then construct the projective modules by giving the projection operator in terms of a normalized vector, following the approach for monopoles on S^2. We find nontrivial projective modules of $\mathbb{C}P^n_N$ labeled by an integer κ, which are interpreted as fuzzy version of the monopole bundles on $\mathbb{C}P^n$ with monopole number κ. Using a suitable differential calculus, we then calculate the field strength over the monopole bundle, or equivalently the first Chern class. We verify explicitly that the usual Chern number c_1 is recovered in the commutative limit.

Finally let us recall that fuzzy spaces arise naturally in string theory, for example as D-branes on group manifolds or as solutions of the IKKT matrix model. In both cases one expects that the low-energy effective action should be given by an induced gauge theory. These gauge theories typically have degrees of freedom which are not tangential as in conventional field theories. As we have seen, such degrees of freedom do arise naturally in the differential calculus on fuzzy spaces.

There are several open questions which deserve further investigations. There exists a somewhat different (but related) formulation of monopoles as solutions of a matrix model on the fuzzy sphere [15]; the extension of this construction to fuzzy $\mathbb{C}P^2$ will be presented elsewhere [31]. Furthermore, fermions have been discussed extensively on the fuzzy sphere, but for $\mathbb{C}P^n$ no fully satisfactory formulation is available. Of course, a generalization to instantons would also be desirable.

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Appendix A. Characteristic equation

A.1. su(3)

Consider the representation

$$V := V_{N\Lambda_1} \otimes V_{\Lambda_1} = V_{(N+1)\Lambda_1} \oplus V_{(N+1)\Lambda_1-\alpha_1} \tag{A.1}$$

of $su(3)$. The operator $X = \sum_a \xi_a \lambda_a$ is an intertwiner on V, where $\lambda_a = 2\pi_{\Lambda_{(1)}}(T_a)$ and $\xi_a = \pi_{N\Lambda_{(1)}}(T_a)$ and T_a are the generators of $su(3)$ which satisfy $[T_a, T_b] = i f_{abc} T_c$. X can be related to the quadratic Casimir of $su(3)$:

$$C_2 = 2\sum_a T_a T_a \tag{A.2}$$

as follows:

$$X = \sum_a \xi_a \lambda_a = 2\sum_a \pi_{N\Lambda_1}(T_a)\pi_{\Lambda_1}(T_a) = \frac{1}{2}(\pi_V(C_2) - \pi_{N\Lambda_1}(C_2) - \pi_{\Lambda_1}(C_2)). \tag{A.3}$$

Recall that the eigenvalues of the quadratic Casimirs on the highest weight representation V_Λ are given by

$$C_2(\Lambda) = (\Lambda, \Lambda + 2\rho), \tag{A.4}$$

where $\rho = \sum \Lambda_i$ is the Weyl vector of $su(3)$, and $(,)$ denotes the Killing form. It follows that the eigenvalue of X on the components $V_{N\Lambda_1+\nu}$ in (A.1) is

$$X = \tfrac{1}{2}(C_2(N\Lambda_1 + \nu) - C_2(N\Lambda_1) - C_2(\Lambda_1)) = (\nu, N\Lambda_1) + (\nu - \Lambda_1, \rho) \tag{A.5}$$

for $\nu = \Lambda_1$ resp. $\nu = \Lambda_1 - \alpha_1$. Using the inner products of the fundamental weights

$$(\Lambda_1, \Lambda_1) = \tfrac{2}{3} = (\Lambda_2, \Lambda_2), \qquad (\Lambda_1, \Lambda_2) = \tfrac{1}{3}, \qquad (\Lambda_2, \rho) = 1 = (\Lambda_1, \rho) \tag{A.6}$$

for $su(3)$, we find the eigenvalues of X as $(\frac{2N}{3}, -\frac{N}{3} - 1)$, hence the characteristic equation of X is

$$\left(X - \frac{2N}{3}\right)\left(X + \frac{N}{3} + 1\right) = 0. \tag{A.7}$$

A.2. $su(n+1)$

For $\mathbb{C}P^n$, consider the representation

$$V := V_{N\Lambda_1} \otimes V_{\Lambda_1} = V_{(N+1)\Lambda_1} \oplus V_{(N+1)\Lambda_1 - \alpha_1} \tag{A.8}$$

of $su(n+1)$. The operator $X = \sum_a \xi_a \lambda_a$ is an intertwiner on V, where $\pi_{\Lambda_1}(T_a) = \frac{1}{2}\lambda_a$ and $\xi_a = \pi_{N\Lambda_1}(T_a)$ and T_a are the generators of $su(n+1)$ which satisfy $[T_a, T_b] = i f_{abc} T_c$. X can again be related to the quadratic Casimir $C_2 = 2\sum_a T_a T_a$ of $su(n+1)$ as follows:

$$X = \sum_a \xi_a \lambda_a = \frac{1}{2}(\pi_V(C_2) - \pi_{N\Lambda_1}(C_2) - \pi_{\Lambda_1}(C_2)). \tag{A.9}$$

Hence the eigenvalue of X on the component $V_{N\Lambda_1+\nu}$ in (A.1) is

$$X = (\nu, N\Lambda_1) + (\nu - \Lambda_1, \rho) \tag{A.10}$$

for $\nu = \Lambda_1$ resp. $\nu = \Lambda_1 - \alpha_1$. The eigenvalues are then $\left(\frac{nN}{n+1}, -\frac{N}{n+1} - 1\right)$, and the characteristic equation of X is given by

$$\left(X - \frac{nN}{n+1}\right)\left(X + \frac{N}{n+1} + 1\right) = 0. \tag{A.11}$$

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